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On support varieties of Auslander–Reiten components

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ABSTRACT

Let $u(L,\chi)$ be the reduced enveloping algebra associated to a finite dimensional restricted Lie algebra $(L,[p])$ and a linear form $\chi \in L^*$. It is shown that a connected component Θ of the stable Auslander–Reiten quiver of $u(L,\chi)$ is of type $\mathbb{Z}[A_\infty]$, whenever its support variety $\mathcal{V}_L(\Theta)$ has dimension ≥ 3. Various applications concerning AR-components of Lie algebras of algebraic groups and the structure of hearts of principal indecomposable $u(L,\chi)$-modules are given.

1. INTRODUCTION AND PRELIMINARIES

In recent work [11] Erdmann has shown that the nonperiodic components of the stable Auslander–Reiten quivers belonging to wild blocks of a modular group algebra are isomorphic to $\mathbb{Z}[A_\infty]$. Accordingly, the Auslander–Reiten theory of group algebras is now very well understood. By contrast, relatively little is known about the AR-quivers of the family $(u(L,\chi))_{\chi\in L^*}$ of reduced enveloping algebras associated to a restricted Lie algebra $(L,[p])$. One main problem in this context is the lack of a comprehensive block theory of enveloping algebras. Thus, aside from the possibility of considering special classes of Lie algebras (cf. [12, 17]), Carlson's geometric approach [9], that was transferred to the theory of restricted Lie algebras by Friedlander and Parshall [20], appears most promising. In this vein, one result was given by Rickard [33], who proved that $u(L,\chi)$ is wild whenever it admits an indecomposable module

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whose support variety has dimension ≥ 3. The principal result of this note, Theorem 2.1, shows that the stable Auslander–Reiten component of such a module has tree class A_∞. In the context of simple Lie algebras our result can be used to identify $\mathfrak{sl}(2)$ as the only algebra whose restricted envelope admits an Auslander–Reiten component of Euclidean tree class. The final section illustrates another application of Theorem 2.1 by showing that the hearts of the principal indecomposable modules of simple Lie algebras $\neq \mathfrak{sl}(2)$ are indecomposable.

Let Λ be a self-injective algebra over a field F. Throughout this paper all modules are assumed to be finite dimensional. The stable Auslander–Reiten quiver $\Gamma_s(\Lambda)$ has the isomorphism types $[M]$ of nonprojective indecomposable Λ-modules as its vertices. There is an arrow $[M] \to [N]$, whenever there exists an irreducible map $M \to N$. We denote the Auslander–Reiten translation of $\Gamma_s(\Lambda)$ by τ and refer the rader to [2] concerning basic properties of AR-quivers.

A map $g : M \to N$ between two indecomposable Λ-modules is called *properly irreducible* if it is irreducible, but not almost split. The following subsidiary results, due to Erdmann (cf. [11, (1.5)]), will play a crucial rôle in the proof of our main result.

Proposition 1.1. *Let $g : B \to C$ be a surjective, properly irreducible homomorphism between two indecomposable modules such that $[C]$ belongs to a component of tree class A_∞^∞ or D_∞. If V is a Λ-module such that there exists a projective module P with $\tau(V) \oplus P \cong V$, then one of the following holds:*

(1) *There is an embedding* $\ker g \hookrightarrow V$, *or*

(2) *every map $\phi : \ker g \to \Omega^{-1}(V)$ which does not factor through a projective module is a monomorphism.*

For future reference we also record the dual version of Proposition 1.1:

Proposition 1.2. *Let $g : C \to B$ be an injective, properly irreducible homomorphism between two indecomposable modules such that $[C]$ belongs to a component of tree class A_∞^∞ or D_∞. If V is a Λ-module such that $\tau^{-1}(V) \oplus P \cong V$ for some projective module P, then one of the following holds:*

(1) *There is a surjection $V \to$* coker g, *or*

(2) *every map $\phi : \Omega(V) \to$* coker g *which does not factor through a projective module is an epimorphism.*

2. SUPPORT VARIETIES OF NONPERIODIC COMPONENTS

Throughout the remainder of the paper $(L, [p])$ is assumed to be a finite dimensional restricted Lie algebra, defined over an algebraically closed field F of characteristic $p > 0$. We fix a linear form $\chi \in L^*$ and consider the stable Auslander–Reiten quiver $\Gamma_s(L, \chi)$ of the reduced enveloping algebra $u(L, \chi)$. By definition, the algebra $u(L, \chi)$ is obtained from the ordinary enveloping algebra $\mathcal{U}(L)$ by factoring out the ideal generated by

$$\{x^p - x^{[p]} - \chi(x)^p 1;\ x \in L\}.$$

In case of $\chi = 0$, $u(L) := u(L, 0)$ is the restricted enveloping algebra of L whose stable Auslander–Reiten quiver will be denoted $\Gamma_s(L)$. According to [38, (V.4.3)] and [18, (1.2)] $u(L, \chi)$ is a Frobenius algebra whose Nakayama automorphism $\mu : u(L, \chi) \to u(L, \chi)$ is determined by

$$\mu(x) = x - tr(\mathrm{ad}\ x)1 \quad \forall x \in L.$$

For any $u(L, \chi)$-module M and every $i \in \mathbb{Z}$, we let $M^{(i)}$ denote the $u(L, \chi)$-module with underlying vector space M and action

$$u \cdot m := \mu^i(u)m \quad \forall u \in u(L, \chi),\ m \in M.$$

Since $u(L, \chi)$ is a Frobenius algebra, general theory (cf. [3, p. 138]) shows that the Auslander–Reiten translation $\tau = \tau_{u(L,\chi)}$ is related to the Heller operator $\Omega_{u(L,\chi)}$ via

$$\tau([M]) = \Omega^2_{u(L,\chi)}([M^{(-1)}]) \quad \forall [M] \in \Gamma_s(L, \chi).$$

To each $u(L, \chi)$-module M we associate the affine variety

$$\mathcal{V}_L(M) := \left\{x \in L;\ x^{[p]} = 0 \text{ and } M|_{u(Fx,\chi|Fx)} \text{ is not projective}\right\} \cup \{0\},$$

which is called the *support variety* of M (cf. [21, p. 1083]). The reader is referred to [20, 21] regarding the cohomological definition and basic properties of support varieties.

Let Θ be a component of $\Gamma_s(L, \chi)$. In analogy with the representation theory of finite groups we have

$$\mathcal{V}_L(M) = \mathcal{V}_L(N)$$

whenever $[M]$, $[N] \in \Theta$ (cf. [15, (5.2)]). Hence we may define the *support variety* of Θ by means of

$$\mathcal{V}_L(\Theta) := \mathcal{V}_L(M)$$

for some $[M] \in \Theta$. Since Θ does not contain a projective module [21, (6.2)] yields $\mathcal{V}_L(\Theta) \neq \{0\}$. The dimension of the variety $\mathcal{V}_L(M)$ is known to coincide with the *complexity* $c_{u(L,\chi)}(M)$ of the $u(L, \chi)$-module M (cf. [20, (3.2)]). It is easily seen (cf. [20, p. 35]) that

$$c_{u(L,\chi)}(M) = \min\left\{c \in \mathbb{N}_0;\ \exists \lambda > 0 \quad \dim_F \Omega^n_{u(L,\chi)}(M) \leq \lambda n^{c-1} \forall n \geq 1\right\}.$$

An indecomposable $u(L, \chi)$-module M is called *periodic* if there exists an $n > 0$ such that $\Omega^n_{u(L,\chi)}(M) \cong M$. If a component Θ contains the type of a periodic module, then all its elements are defined by periodic modules (cf. [22]), and we will refer to this fact by saying that Θ is *periodic*. It was shown in [15] that the periodic components are finite or infinite tubes of rank $r \in \{1, p\}$. In view of [19, §2], [20, §3, §4], and [21, §6, §7] the arguments of [4, (5.10.4)] may be adopted verbatim to show that a nonprojective indecomposable $u(L, \chi)$-module is periodic if and only if $c_{u(L,\chi)}(M) = 1$. Consequently, by [20, (4.3)], the periodic components of $\Gamma_s(L,\chi)$ are those, whose support varieties are lines.

We shall call a component Θ *Euclidean* if Θ has Euclidean tree class. According to [12, Theorem 1] or [15, (5.7)] a nonperiodic component is either Euclidean, or of tree class A_∞^∞, or isomorphic to $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[D_\infty]$.

Theorem 2.1. *Let $\Theta \neq \mathbb{Z}[A_\infty]$ be a nonperiodic component of $\Gamma_s(L,\chi)$. Then* $\dim \mathcal{V}_L(\Theta) = 2$.

Proof. By our foregoing remarks we have

$$\dim \mathcal{V}_L(\Theta) \geq 2.$$

Let $[B]$ be an element of Θ, and suppose first Θ to be Euclidean. In virtue of [13, (3.2)] every simple $u(L,\chi)$-module S gives rise to subadditive functions

$$d_S^i : \Theta \to \mathbb{N}_0;\ [M] \mapsto \dim_F \mathrm{Ext}^i_{u(L,\chi)}(M,S) \quad 1 \leq i \leq 2$$

that fail to be additive at only finitely many vertices. The remark of [40, p. 105] then implies that the rate of growth of the sequence $(\mathrm{Ext}^n_{u(L,\chi)}(B,S))_{n\geq 1}$ is bounded by 2. Owing to [1, (5.4)] this entails $c_{u(L,\chi)}(B) \leq 2$, and [20, (3.2)] yields

$$\dim \mathcal{V}_L(\Theta) = c_{u(L,\chi)}(B) \leq 2.$$

Alternatively, Θ has tree class A_∞^∞ or D_∞, and we can choose $[B] \in \Theta$ to have two successors and two predecessors.

Given an element $x \in \mathcal{V}_L(F) \setminus \{0\}$, we denote the unique simple $u(Fx,\chi|_{Fx})$-module by F_χ and consider the module

$$M_x := u(L,\chi) \otimes_{u(Fx,\chi|_{Fx})} F_\chi.$$

Since $\Omega^2_{u(Fx,\chi|_{Fx})}(F_\chi) \cong F_\chi \cong \Omega^{-2}_{u(Fx,\chi|_{Fx}}(F_\chi)$ there exist projective modules P, Q such that $\Omega^2_{u(L,\chi)}(M_x) \oplus P \cong M_x \cong \Omega^{-2}_{u(L,\chi)}(M_x) \oplus Q$. Moreover, as $u(Fx,\chi|_{Fx})$ surjects onto F_χ and F_χ injects into $u(Fx,\chi|_{Fx})$, M_x is a factor module as well as a submodule of $u(L,\chi)$. Consequently,

$$\max\left\{\dim_F M_x,\ \dim_F \Omega_{u(L,\chi)}(M_x),\ \dim_F \Omega^{-1}_{u(L,\chi)}(M_x)\right\} \leq k,$$

where $k := \dim_F u(L,\chi)$.

Since $\mu|_{u(Fx,\chi|_{Fx})} = \mathrm{id}_{u(Fx,\chi|_{Fx})}$ the map

$$\psi : M_x \to M_x;\ \psi(a \otimes 1) = \mu(a) \otimes 1$$

is readily seen to induce an isomorphism $M_x \cong M_x^{(1)}$. Thus, the above isomorphisms yield

$$\tau(M_x) \oplus P \cong M_x \cong \tau^{-1}(M_x) \oplus Q.$$

Now let $g : B \to C$ be irreducible. By choice of $[B]$ this map is properly irreducible. If g is surjective, then $A := \ker g$ is not a projective module, so that there is $x \in \mathcal{V}_L(A) \setminus \{0\}$. According to [18, p. 158f] $(u(L,\chi) : u(Fx,\chi|_{Fx}))$ is a

Frobenius extension of first kind, so that general properties of Frobenius Extensions (cf. [30, p. 96f]) imply

$$M_x \cong \mathrm{Hom}_{u(Fx,\chi|_{Fx})}(u(L,\chi), F_\chi).$$

Since A is not $u(Fx, \chi|_{Fx})$-projective, Frobenius reciprocity yields

$$\mathrm{Ext}^1_{u(L,\chi)}(A, M_x) \cong \mathrm{Ext}^1_{u(Fx,\chi|_{Fx})}(A, F_\chi) \neq (0).$$

Proposition 1.1 now shows $\dim_F A \leq k$, so that

$$\dim_F B \leq \dim_F C + k.$$

If g is injective, then $A := \operatorname{coker} g$ is not projective and there is $x \in \mathcal{V}_L \setminus \{0\}$. Hence A is not $u(Fx, \chi|_{Fx})$-injective, and

$$\dim_F \mathrm{Ext}^1_{u(L,\chi)}(M_x, A) = \dim_F \mathrm{Ext}^1_{u(Fx,\chi|_{Fx})}(F_\chi, A) \neq (0).$$

Thus, Proposition 1.2 provides the estimate

$$\dim_F C \leq \dim_F B + k$$

in this case.

By choice of $[B]$ there exists a walk $\tau(B) \to C \to B$ in Θ whose arrows are given by properly irreducible maps. Consequently, the above observations show that

$$\dim_F \Omega^2_{u(L,\chi)}(B) = \dim_F \tau(B) \leq \dim_F B + 2k.$$

Repeated application of τ thus implies

$$\dim_F \Omega^{2n}_{u(L,\chi)}(B) \leq \dim_F B + 2nk.$$

By considering the vertex $[\Omega_{u(L,\chi)}(B)] \in \Omega_{u(L,\chi)}(\Theta) \cong \Theta$, we conclude the existence of $q > 0$ such that

$$\dim_F \Omega^n_{u(L,\chi)}(B) \leq qn \quad \forall n \geq 1.$$

This proves $c_{u(L,\chi)}(B) \leq 2$, so that

$$\dim \mathcal{V}_L(\Theta) \leq 2,$$

as desired. □

Remark. The converse of Theorem 2.1 fails, that is, components of tree class A_∞ may have two-dimensional varieties. Consider for example the restricted enveloping algebra $u(L)$ of the two-dimensional strongly abelian Lie algebra $L = Fx \oplus Fy$, whose Lie product and p-map are trivial. Let Θ be the component containing the trivial module, so that $\mathcal{V}_L(\Theta) = L$ is two-dimensional. According to [12, Theorem 2, Theorem 3] we have $\Theta \cong \mathbb{Z}[A_\infty]$ for $p \geq 3$. If $p = 2$, then $u(L)$ is the Kronecker algebra and $\Theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$.

Corollary 2.2. *Suppose that* $\dim \mathcal{V}_L(F) \geq 3$, *and let* $\Theta \neq \mathbb{Z}[A_\infty]$ *be a component of* $\Gamma_s(L,\chi)$. *Then* $\dim_F M \equiv 0 \bmod (p)\ \forall [M] \in \Theta$.

Proof. According to Theorem 2.1 and our foregoing remarks concerning periodic components, the variety $\mathcal{V}_L(\Theta)$ is properly contained in $\mathcal{V}_L(F)$. Thus, if $x \in \mathcal{V}_L(F) \setminus \mathcal{V}_L(\Theta)$, then each $[M] \in \Theta$ is given by a projective $u(Fx, \chi|_{Fx})$-module. Since the latter algebra is local and of dimension p, the assertion follows. $\square$

3. AR-COMPONENTS FOR SIMPLE LIE ALGEBRAS

In this section we shall apply Theorem 2.1 to Lie algebras of almost simple algebraic groups. In order to exploit the correspondence between subalgebras and subgroups, we assume for the remainder of this section that $p \geq 5$. The reader is referred to [27, 37] concerning the undefined terminology.

Let G be an algebraic group with Lie algebra $L = \mathrm{Lie}(G)$. We let G operate on L via the adjoint representation $\mathrm{Ad} : G \to GL(L)$. Given a $u(L)$-module M and an element $g \in G$, the module ${}^gM$ is obtained from M by twisting the action by $\mathrm{Ad}(g^{-1})$:

$$x \cdot m := \mathrm{Ad}(g^{-1})(x)m \quad \forall x \in L,\ m \in M.$$

We shall refer to M as $\mathrm{Ad}(G)$-*stable* if ${}^gM \cong M\ \forall g \in G$. Note that every G-module gives rise to an $\mathrm{Ad}(G)$-stable $u(L)$-module.

Proposition 3.1. *Let G be a connected, reductive linear algebraic group with Lie algebra L, M an* $\mathrm{Ad}(G)$*-stable $u(L)$-module. Then the following statements hold:*

(1) $\dim \mathcal{V}_L(M) \neq 1$.

(2) *If* $\dim \mathcal{V}_L(M) = 2$, *then there exists an almost simple, normal subgroup $H \subset G$ such that $\mathrm{Lie}(H) \cong \mathfrak{sl}(2)$ and $L = \mathrm{Lie}(H) \oplus J$ is a direct sum of p-ideals.*

Proof. We may assume that $\mathcal{V}_L(M) \neq \{0\}$. Since M is $\mathrm{Ad}(G)$-stable, the conical variety $\mathcal{V}_L(M)$ is readily seen to be invariant under the adjoint representation. The resulting action $G \times \mathcal{V}_L(M) \to \mathcal{V}_L(M)$ determines an operation

$$(g, [x]) \mapsto [\mathrm{Ad}(g)(x)]$$

of G on the projective variety $\mathrm{Proj}(\mathcal{V}_L(M))$ of $\mathcal{V}_L(M)$. Let $B \subset G$ be a Borel subgroup of G. According to Borel's fixed point theorem there exists x_0 in $\mathcal{V}_L(M) \setminus \{0\}$ such that $B \cdot [x_0] = [x_0]$. Thus, the isotropy subgroup $G_0 \subset G$ of $[x_0]$ is a parabolic subgroup of G and contains a maximal torus $T \subset G$.

(1) If $\dim \mathcal{V}_L(M) = 1$, then $\mathrm{Proj}(\mathcal{V}_L(M))$ is finite. Since G is connected, it follows that $G = G_0$. Thus, Fx_0 is Ad-invariant, and therefore a p-nilpotent ideal of L. According to [26, (11.8)] the Lie algebra L does not possess any non-trivial p-nilpotent ideals, a contradiction.

(2) General theory provides an injective, G-equivariant morphism $G/G_0 \hookrightarrow \mathrm{Proj}(\mathcal{V}_L(M))$. Consequently,

$$\dim G - \dim G_0 = \dim G/G_0 \leq 1.$$

In view of (1) we may assume that $\dim G_0 = \dim G - 1$. Using the arguments of

[7, Proposition 13.13] we obtain a surjective homomorphism $\varphi : G \to PGL_2$ of algebraic groups. It follows that there exists an almost simple constituent $H \subset G$ such that $\varphi(H) = PGL_2$. Consequently, $\dim H = 3$ so that $\mathrm{Lie}(H) \cong s\ell(2)$ is a p-ideal of L. Since $\mathrm{Lie}(H)$ is complete, i.e., centerless with all derivations being inner, L is the direct sum of $\mathrm{Lie}(H)$ and its centralizer. □

Theorem 3.2. *Let $L = Lie(G)$ be the Lie algebra of a connected, almost simple linear algebraic group, $\Theta \subset \Gamma_s(L)$ a component. Then the following statements hold:*

(1) *If $L \ncong s\ell(2)$ and Θ contains the type of an* Ad(G)*-stable module, then $\Theta \cong \mathbb{Z}[A_\infty]$. Moreover, every module belonging to Θ is* Ad(G)*-stable.*

(2) *If $L \ncong s\ell(2)$ and Θ contains the type of a simple module, then $\Theta \cong \mathbb{Z}[A_\infty]$.*

(3) *If Θ is Euclidean, then $L \cong s\ell(2)$ and $\Theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$.*

Proof. (1) According to Proposition 3.1, we have $\dim \mathcal{V}_L(\Theta) \geq 3$, and the first assertion now follows directly from Theorem 2.1.

Since $\det(\mathrm{Ad}(g)) = 1$ for every $g \in G$, we have $\mathrm{tr} \circ \mathrm{ad} = 0$, proving that $u(L)$ is symmetric. Given $g \in G$ the equivalence $M \mapsto {}^gM$ induces an automorphism on $\Gamma_s(L)$ that leaves Θ invariant. Thus, the induced automorphism on the tree A_∞ coincides with the identity map and there exists $n \in \mathbb{Z}$ such that

$$(*) \qquad {}^gM \cong \Omega^{2n}_{u(L)}(M) \quad \forall [M] \in \Theta.$$

Replacing g by g^{-1} if necessary, we may assume $n \geq 0$. We obtain

$$\dim_F \Omega^{2kn}_{u(L)}(M) = \dim_F {}^{g^k}M = \dim_F M \quad \forall k \geq 0,$$

which, in case of $n \neq 0$, shows that $(\dim_F \Omega^m_{u(L)}(M))_{m \geq 0}$ is bounded. Since Θ does not contain any periodic modules, we have $n = 0$. Consequently, $(*)$ shows that every module belonging to Θ is $\mathrm{Ad}(G)$-stable.

(2) According to general theory (cf. [28, p. 220]), every simple $u(L)$-module S is $\mathrm{Ad}(G)$-stable so that (1) applies.

(3) Let Θ be Euclidean. Owing to Theorem 2.1 we have $\dim \mathcal{V}_L(M) = 2$. Moreover, the analogue of [40, Theorem A] shows that Θ is attached to a principal indecomposable module. Consequently, the Euclidean component $\Omega_{u(L)}(\Theta)$ contains the vertex of a simple module, and (2) implies $L \cong s\ell(2)$. It readily follows from Pollack's work (cf. [32]) that the Euclidean components of $\Gamma_s(s\ell(2))$ are isomorphic to $\mathbb{Z}[\tilde{A}_{1,2}]$. □

Corollary 3.3. *Let L be a classical simple Lie algebra, $\Theta \subset \Gamma_s(L)$ a component. Then the following statements hold:*

(1) *If Θ is Euclidean, then $L \cong s\ell(2)$ and $\Theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$.*

(2) *If $L \ncong s\ell(2)$ and Θ is not periodic, then $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[D_\infty]$ or $\mathbb{Z}[A_\infty^\infty]$. Moreover, the nonprojective simple $u(L)$-modules belong to components of type $\mathbb{Z}[A_\infty]$.*

Proof. According to [26, (5.4)] there exists a connected, almost simple linear algebraic group G and a surjective homomorphism $\pi : \mathrm{Lie}(G) \to L$ such that $\ker \pi = C(\mathrm{Lie}(G))$ is a torus of dimension ≤ 1 and $[\mathrm{Lie}(G), \mathrm{Lie}(G)] = \mathrm{Lie}(G)$. This readily implies that the induced morphism $\pi : \mathcal{V}_{\mathrm{Lie}(G)}(F) \to \mathcal{V}_L(F)$ is injective.

Let Ψ be a component of $\Gamma_s(L)$ containing a simple module S. Since S is a simple $\mathrm{Lie}(G)$-module, it is $\mathrm{Ad}(G)$-stable. Using the fact that $\ker \pi$ is a torus, we obtain the inclusion $\pi(\mathcal{V}_{\mathrm{Lie}(G)}(S)) \subset \mathcal{V}_L(S)$, so that

$$\dim \mathcal{V}_{\mathrm{Lie}(G)}(S) \leq \dim \mathcal{V}_L(S).$$

If $L \not\cong s\ell(2)$, then $\mathrm{Lie}(G) \not\cong s\ell(2)$. Thus, from a consecutive application of Proposition 3.1 and Theorem 2.1, we obtain $\Psi \cong \mathbb{Z}[A_\infty]$.

Now suppose that Θ is either Euclidean or $\Theta \cong \mathbb{Z}[A_\infty^\infty]/\Pi$, where $\{1\} \neq \Pi \subset \mathrm{Aut}(\mathbb{Z}[A_\infty^\infty])$ is an admissible group of automorphisms. By combining [40, Theorem A] with a result by Butler and Ringel [8, p. 155], we conclude that Θ is attached to a principal indecomposable module. Thus, the component $\Psi = \Omega_{u(L)}(\Theta)$ contains the vertex of a simple $u(L)$-module. By our observations above this implies $L \cong s\ell(2)$, and we have established (1).

By applying [15, (5.7)] and (1) consecutively, we see that either $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[D_\infty]$ or Θ has tree class A_∞^∞. In the latter case Riedtmann's structure theorem (cf. [3, (4.15.6)]) shows that $\Theta \cong \mathbb{Z}[A_\infty^\infty]/\Pi$, where $\Pi \subset \mathrm{Aut}(\mathbb{Z}[A_\infty^\infty])$ is an admissible group of automorphisms. It now follows from the arguments above that $\Pi = \{1\}$. □

The class of finite dimensional algebras over F can be divided into the subclasses of representation finite, tame, and wild algebras. If an algebra belongs to the first two classes, then there is hope for classifying its indecomposable modules. We refer the reader to [10] for the precise definitions.

Corollary 3.4. *Let $L \not\cong s\ell(2)$ be a classical simple Lie algebra. If $\mathcal{B} \subset u(L)$ is a block not associated to the Steinberg module, then $\mathcal{B}$ is wild.*

Proof. We adopt the notation of the proof of Corollary 3.3. Let S be a simple $\mathcal{B}$-module. Since $\mathcal{B}$ is not associated to the Steinberg module, S is not projective. Hence S is not projective, when considered a $u(\mathrm{Lie}(G))$-module. Since $\mathrm{Lie}(G) \not\cong s\ell(2)$ and S is $\mathrm{Ad}(G)$-stable, Proposition 3.1 in conjunction with the arguments of Corollary 3.3 implies

$$\dim \mathcal{V}_L(S) \geq \dim \mathcal{V}_{\mathrm{Lie}(G)}(S) \geq 3.$$

The assertion now follows directly from [33, Theorem 2]. □

Remark. Neither Corollary 3.3 nor Corollary 3.4 retain their validity in the more general setting of reduced enveloping algebras. According to [17, (5.4)] the simple Lie algebras $L = s\ell(4)$, $o(5)$ possess linear forms $\chi \in L^*$ such that $u(L, \chi)$ has tame blocks whose AR-quivers have Euclidean components.

We now turn to the representation theory of the restricted Lie algebras of Cartan type. The interested reader may consult [38, Chapter 4] for the definition and elementary properties of these algebras. For our purposes it will suffice to list the following facts. The algebras are divided into four families $(W(n))_{n\geq 1}$, $(S(n))_{n\geq 3}$, $(H(2r))_{r\geq 1}$, and $(K(2r+1))_{r\geq 1}$. Each algebra L comes fitted with a restricted $\mathbb{Z}$-grading, i.e., a vector space decomposition $L = \oplus_{i=-t}^{s} L_i$ $(s, t > 0)$ satisfying

$$[L_i, L_j] \subset L_{i+j}; \quad L_i^{[p]} \subset L_{pi}.$$

In addition, we have $W(n)_0 \cong g\ell(n)$, $S(n)_0 \cong s\ell(n)$, $H(2r)_0 \cong sp(2r)$, and $K(2r+1)_0 \cong sp(2r) \oplus Fz$, where Fz is a central element of $K(2r+1)_0$.

Corollary 3.5. *Let L be a restricted Lie algebra of Cartan type. Then $\Gamma_s(L)$ does not possess any Euclidean components. Moreover, any nonperiodic component of $\Gamma_s(L)$ is isomorphic to $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[D_\infty]$, or $\mathbb{Z}[A_\infty^\infty]$, and every simple module is contained in a component of type $\mathbb{Z}[A_\infty]$.*

Proof. By the results of [25, §3] the algebra $u(L)$ has one block. Since $u(L)$ is not semisimple (cf. [23]), it follows that every simple $u(L)$-module is nonprojective. Proceeding as in Corollary 3.3, we consider a component Θ containing a simple $u(L)$-module S. According to [35, (1.5)] the module S admits a $\mathbb{Z}$-grading

$$S = \bigoplus_{j=0}^{q} S_j$$

such that $L_i \cdot S_j \subset S_{i+j}$. In particular, each subspace is L_0-invariant and, thanks to [35, (1.1)], S_0 is a simple L_0-module. General properties of support varieties now imply

$$\mathcal{V}_{L_0}(S_0) \subset \mathcal{V}_{L_0}(S) \subset \mathcal{V}_L(S).$$

If L is of type H or K, then S_0 is a simple $sp(2r)$-module, and Proposition 3.1 shows that $\dim \mathcal{V}_L(S) \geq 3$ unless $L \cong H(2)$ or $L \cong K(3)$. In these cases, this estimate follows from [29, (5.1)]. For $L = S(n)$, $W(n)$ the module S_0 is $\mathrm{Ad}(SL(n))$-stable and Proposition 3.1 yields $\dim \mathcal{V}_L(S) \geq 3$ unless $n \leq 2$. However, these cases are covered by [29, (5.1), (5.4)].

As a result, we have $\dim \mathcal{V}_L(S) \geq 3$, and Theorem 2.1 shows that $\Theta \cong \mathbb{Z}[A_\infty]$. The proof may now be concluded by arguing as in Corollary 3.3. □

4. HEARTS OF PRINCIPAL INDECOMPOSABLE MODULES

In view of the results of the preceding section one is interested in the position of a simple module within its Auslander–Reiten component. We say that a module *M lies at an end of a component Θ* if $[M]$ has exactly one predecessor. The example of $u(s\ell(2))$ shows that simple modules are not necessarily located at ends of components. However, as we shall see below, this does not happen for the other simple Lie algebras or for Lie algebras of solvable algebraic groups.

In all these cases the structure of the *heart* $H(P) := \mathrm{rad}(P)/\mathrm{soc}(P)$ of a principal indecomposable module P is related to the representation type of the algebra $u(L)$ in the sense that decomposability of $H(P)$ is equivalent to P belonging to a tame block.

Proposition 4.1. *Let S be a simple $u(L, \chi)$-module such that the component Θ containing $[S]$ is isomorphic to $\mathbb{Z}[A_\infty]$. Then $[S]$ is located at an end of Θ, and the heart $H(P)$ of the projective cover P of S is indecomposable. Moreover, $[S]$ is the only simple vertex of Θ.*

Proof. The arrows of a component $\Theta = \mathbb{Z}[A_\infty]$ either point towards the end or towards infinity. Accordingly, for any vertex $[M]$ there exists exactly one sectional path from an end vertex to $[M]$. The number of vertices on that path is called the *quasilength* $\mathrm{ql}([M])$ of M (cf. [36, p. 479]).

Suppose the proposition to be false. For each component $\Theta \cong \mathbb{Z}[A_\infty]$ we let $\mathcal{S}(\Theta) := \{[S]; S \text{ simple}, \mathrm{ql}([S]) \geq 2\}$ and, in case $\mathcal{S}(\Theta) \neq \emptyset$, we define

$$q_\Theta := \min\{\mathrm{ql}([S]); [S] \in \mathcal{S}(\Theta)\}.$$

Pick Θ such that $q_\Theta \geq 2$ is minimal. It follows from [2, (I.5.7)] that parallel maps within a nonprojective mesh of Θ have isomorphic kernels and cokernels. Let $[S] \in \Theta$ be a vertex of quasilength $\mathrm{ql}([S]) = q_\Theta$ and suppose that $q_\Theta \geq 3$. Since every irreducible map terminating (originating) in S is surjective (injective), arguments similar to those of [5, p. 250f] show that there are two projective meshes whose vertices have quasilength $< q_\Theta$. By considering the component $\Omega(\Theta) \cong \Theta$ we obtain from the minimality of q_Θ, that the corresponding simple modules have quasilength 1. Thus, the τ-orbit of modules of quasilength 1 contains two simple modules and there exists a simple module $[T] \in \Omega(\Theta)$ and $n > 0$ such that $\Omega^{2n}(T)$ is simple. Accordingly, we have $\dim_F \mathrm{Ext}^{2n}_{u(L,\chi)}(T, T) \leq 1$, and [15, (2.1)] implies the periodicity of T, a contradiction.

Consequently, $q_\Theta = 2$ and there is $[M] \in \Theta$ such that $[S]$ is the only successor of $[M]$. There results an almost split sequence

$$(0) \to M \to S \oplus Q \to N \to (0),$$

where Q is a (necessarily nonzero) principal indecomposable module. Since the above sequence is isomorphic to the standard sequence, we have $H(Q) \cong S$. By the same token, it now follows that $[N] \in \Theta$ is a vertex of length 2 such that $\Omega^2_{u(L,\chi)}(N)$ has length 2. According to [17, (3.1)] this yields the periodicity of N, a contradiction.

Hence the simple vertices are located at the end of Θ, and the argument at the end of the first paragraph shows that Θ possesses exactly one simple vertex. □

We shall apply the foregoing result to Lie algebras of almost simple and solvable algebraic groups. In order to formulate our main result for the latter in a slightly more general setting we recall that a restricted Lie algebra $(L, [p])$ is said to be *supersolvable* if its first derived algebra $[L, L]$ is nilpotent. The reader is referred to [24] concerning basic facts from relative homological algebra.

Theorem 4.2. *Suppose that $p \geq 3$. Let L be supersolvable, $\Theta \subset \Gamma_s(L, \chi)$ a non-periodic component containing a one-dimensional module F_λ. Then $\Theta \cong \mathbb{Z}[A_\infty]$, and F_λ is located at an end of Θ.*

Proof. Let $\mathcal{B} \subset u(L, \chi)$ be the block of the one-dimensional module F_λ. According to [16, (2.2)] $\mathcal{B}$ is isomorphic to $u(L/T(L))$, where $T(L)$ is the maximal toral ideal of L. It readily follows from [38, (II.3.4)] and [14, (4.3)] that $L/T(L)$ is the semidirect sum of its p-unipotent radical and a maximal torus. Accordingly, we may assume without loss of generality that $\mathcal{B} = u(L)$, and

$$L = T \oplus \mathrm{rad}_p(L),$$

for some maximal torus $T \subset L$. Let J and $u(\mathrm{rad}_p(L))^+$ be the Jacobson radical of $u(L)$ and the augmentation ideal of $u(\mathrm{rad}_p(L))$, respectively. According to [38, (I.3.7)] the ideal $u(L)u(\mathrm{rad}_p(L))^+ = u(\mathrm{rad}_p(L))^+u(L)$ is nilpotent, while $u(L)/(u(L)u(\mathrm{rad}_p(L))^+) \cong u(T)$ is semisimple. Consequently, $J = u(L)u(\mathrm{rad}_p(L))^+$ and it follows that every projective cover $P \to M$ of $u(L)$-modules defines a projective cover $P|_{u(\mathrm{rad}_p(L))} \to M|_{u(\mathrm{rad}_p(L))}$. Consequently, $\Omega^2_{u(L)}(F_\lambda)|_{u(\mathrm{rad}_p(L))} \cong \Omega^2_{u(\mathrm{rad}_p(L))}(F_\lambda|_{u(\mathrm{rad}_p(L))})$, and

$$\tau_{u(L)}(F_\lambda)|_{u(\mathrm{rad}_p(L))} \cong \tau_{u(\mathrm{rad}_p(L))}(F_\lambda|_{u(\mathrm{rad}_p(L))}).$$

Let

$$(0) \to \tau_{u(L)}(F_\lambda) \to X_\lambda \to F_\lambda \to (0)$$

be the almost split sequence terminating in F_λ. According to [15, (2.4)] the extension $u(L) : u(\mathrm{rad}_p(L))$ is separable, so that the module F_λ is $(u(L), u(\mathrm{rad}_p(L))$-projective (cf. [31, p. 122ff]). In particular, the sequence

$$(0) \to \tau_{u(L)}(F_\lambda)|_{u(\mathrm{rad}_p(L))} \to X_\lambda|_{u(\mathrm{rad}_p(L))} \to F_\lambda|_{u(\mathrm{rad}_p(L))} \to (0)$$

does not split, and [2, (V.2.4)] ensures that this sequence is the almost split sequence terminating in $F_\lambda|_{u(\mathrm{rad}_p(L))}$. A consecutive application of [12, Theorem 3] and Proposition 4.1 now shows that $F_\lambda|_{u(\mathrm{rad}_p(L))}$ is located at an end of a component of tree class A_∞. Consequently, the module $X_\lambda|_{u(\mathrm{rad}_p(L))}$ and, a fortiori X_λ, is indecomposable. Thus, F_λ is located at an end of Θ. Since L is supersolvable, [17, (5.1)] implies that Θ has tree class A_∞ or D_∞.

In the latter case we have $\Theta \cong \mathbb{Z}[D_\infty]$. Since L is supersolvable, the block $\mathcal{B}$ is transitive in the sense that there is an elementary abelian p-group $\mathcal{G}$ of automorphisms of $\mathcal{B}$ such that every simple module of $\mathcal{B}$ is obtained from a given one by twisting the action by an element of $\mathcal{G}$. In particular, all principal indecomposable modules of $\mathcal{B}$ have the same length $\ell_{\mathcal{B}}$, which according to [16, (2.10)] is a p-power. Since the elements of $\mathcal{G}$ commute with the Nakayama automorphism, $\mathcal{G}$ operates on $\Gamma_s(\mathcal{B})$. Consider the component

$\Psi := \Omega_{u(L)}(\Theta) \cong \Theta$. The projective cover P_λ of F_λ is attached to Ψ. Suppose that the principal indecomposable module $Q \cong {}^gP_\lambda$ $(g \in \mathcal{G})$ is attached to Ψ. Then ${}^g\Psi = \Psi$ and g induces an automorphism of order $\in \{1, p\}$ on Ψ. Since $\mathbb{Z}[D_\infty]$ is easily seen not to possess any automorphisms of finite order ≥ 3, it follows that $Q \cong P_\lambda$. As F_λ is located at an end of Θ, the vertex $[\mathrm{rad}(P_\lambda)] \in \Psi$ has the same property. Thus, $H(P_\lambda)$ has three predecessors, and there exists a vertex $[X] \neq [\mathrm{rad}(P_\lambda)]$ such that $H(P_\lambda)$ is the only successor of $[X]$. By the above there results an almost split sequence

$$(0) \to X \to H(P_\lambda) \to Y \to (0).$$

Since all principal indecomposable $\mathcal{B}$-modules have the same length, we obtain, letting $\ell(M)$ denote the length of a module,

$$\ell(X) = \ell(\Omega^2_{u(L)}(Y)) \equiv \ell(Y) \mod (\ell_{\mathcal{B}}).$$

Thus, $\ell(X) = \ell(Y)$ so that

$$\ell_{\mathcal{B}} \equiv \ell(H(P_\lambda)) \equiv 0 \mod (2),$$

a contradication. This shows that $\Theta \cong \mathbb{Z}[A_\infty]$. □

Corollary 4.3. *Suppose that $p \geq 3$ and let G be a connected, solvable algebraic group. Then every component of $\Gamma_s(Lie(G))$ containing a simple module has tree class A_ℓ or A_∞, and the hearts of the nonsimple principal indecomposable $u(Lie(G))$-modules are indecomposable.*

Proof. Let U_G and T_G denote the unipotent radical and a maximal torus of G, respectively. According to [37, Theorem 6.11] the product map induces an isomorphism

$$G \cong U_G \times T_G$$

of varieties. Differentiation then shows that

$$\mathrm{Lie}(G) = \mathrm{rad}_p(\mathrm{Lie}(G)) \oplus T,$$

where $T = \mathrm{Lie}(T_G)$ is a maximal torus of $\mathrm{Lie}(G)$ (cf. [26, (11.11), (13.2)]). It follows that every simple $u(\mathrm{Lie}(G))$-module is one-dimensional, so that the first statement is a direct consequence of Theorem 4.2 in case the component is not periodic. Alternatively, a consecutive application of [15, (3.2)] and [15, (5.1)] shows that the component has tree class A_ℓ.

Suppose the component Θ to be a attached to the principal indecomposable module P. Then $\Omega^{-1}(\Theta)$ contains a simple module. According to the above Θ has tree class A_∞ or A_ℓ. In the latter case $\mathrm{rad}(P)$ is located at an end of Θ, while Theorem 4.2 shows this to hold in the former. In particular, $H(P)$ is indecomposable. □

Remark. According to [16, (2.2)] every block of $u(\mathrm{Lie}(G))$ is the restricted enveloping algebra of a suitable factor algebra of Lie(G), which in virtue of Voigt's Theorem (cf. [39, Theorem 4]) is not tame.

Theorem 4.4. *Let $p > 7$ and suppose that $(L, [p])$ is simple. If $L \not\cong \mathfrak{sl}(2)$, then the hearts of the nonsimple principal indecomposable $u(L)$-modules are indecomposable.*

Proof. According to the Block–Wilson classification theorem [6], the algebra L is either classical or of Cartan type. The results of §3 show that, for $L \not\cong \mathfrak{sl}(2)$, the type of any nonprojective simple $u(L)$-module is contained in a component of $\Gamma_s(L)$ of the form $\mathbb{Z}[A_\infty]$. Thus, Proposition 4.1 yields the result. □

Example. Let $p \geq 7$ and consider the p-dimensional Witt algebra $W(1)$ with its natural filtration $(W(1)_{(i)})_{i \geq -1}$ (cf. [38, Chapter 4] for details). Given a linear form $\chi \in W(1)^*$, we put

$$r(\chi) := \min\{i \geq -1; \quad \chi(W(1)_{(i)}) = (0)\}.$$

If $0 \leq r(\chi) \leq p-5$, then [16, (5.1)] ensures that the support variety of every simple $u(W(1), \chi)$-module has dimension ≥ 3. By results of [29], the same holds for $r(\chi) = -1$. Hence, for $r(\chi) \leq p-5$, the simple $u(W(1), \chi)$-modules are located at ends of components of type $\mathbb{Z}[A_\infty]$. More detailed arguments show that this is also true for $r(\chi) = p-4$. By [16, (5.2)] $u(W(1), \chi)$ is a Nakayama algebra in the remaining cases, and $\Gamma_s(W(1), \chi) \cong \mathbb{Z}[A_{p-1}]/(\tau)$ for $r(\chi) \neq p-1$, while $\Gamma_s(W(1), \chi) = \emptyset$ for $r(\chi) = p-1$.

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